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Viscous sheets advancing over dry beds

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The unsteady creeping motion of a thin sheet of viscous liquid as it advances over a gently sloping dry bed is examined. Attention is focused on the motion of the leading edge under various influences and four problems are discussed. In the first problem the fluid is travelling down an open channel formed by two straight parallel retaining walls placed perpendicular to an inclined plane. When the channel axis is parallel to the fall line there is a progressive-wave solution with a straight leading edge, but inclination of the axis generates distortions and these are calculated. In the second problem a sheet with a straight leading edge travelling over an inclined plane penetrates a region where the bed is uneven, and the subsequent deformation of the leading edge is followed. The third problem considers the flow down an open channel of circular cross-section (a partially filled pipe) and the time-dependent shape of the leading edge is calculated. The fourth problem is that of flow down an inclined plane with a single curved retaining wall. These problems are all analysed by assuming that a length characteristic of the geometry is large compared with the fluid depth divided by the bed slope, and all the solutions display extreme sensitivity to the data.

1. Introduction and governing equations

The motion of thin viscous sheets down inclined surfaces has received a great deal of attention, but this has been almost exclusively concerned with two-dimensional flows and especially with wave motion associated with the free surface. A recent reference, from which other references may be obtained, is the paper by Lin (1974). Three-dimensional problems, characterized by a leading edge that marks the boundary between the fluid and the dry bed, and whose shape fluctuates during the course of the motion, have received very little attention. The fluctuations can occur because of the presence of obstacles on the bed, unevenness in the bed surface, or an initial shape incompatible with steady motion. The present paper is concerned with a theory of such motions.

It may be easily confirmed by the kitchen experimenter that generation of a straight leading edge that remains straight during the course of the motion is exceedingly difficult if not impossible. It is tempting to attribute this to instability (in the usual sense that infinitesimal disturbances grow until they become significant) and certainly for large bed slopes there is reason to believe that this is the source of the difficulty.† However, the special problems analysed in the present paper suggest that for small

† If the slope of a plane over which syrup is spreading with an approximately straight leading edge is abruptly increased from a very small value to an $O(1)$ value, the leading edge forms long, strikingly uniform fingers.

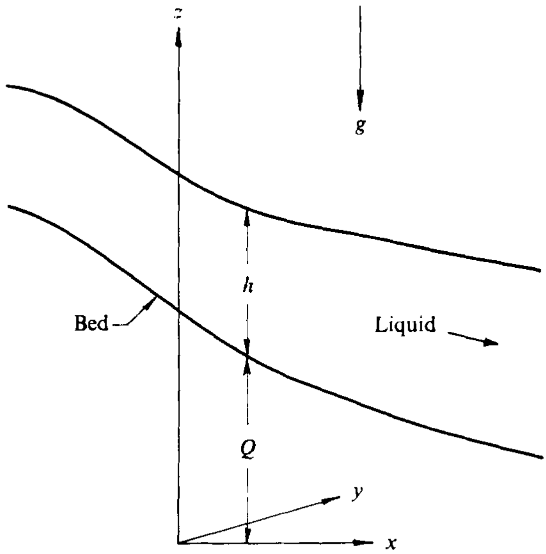

FIGURE 1. Co-ordinate system.

bed slopes and insignificant inertia the difficulty arises because of extreme sensitivity to the data. For example, small non-uniformities in a retaining wall can generate large distortions of the leading edge. Moreover, non-uniformities can generate distortions that fail to disappear after their source has been left far behind. As an example, unevenness in the slope of the bed generates distortions that persist long after the leading edge has passed on to a region of the bed that is perfectly flat. These two effects, data sensitivity and persistence, could easily account for the experimental difficulty and constitute the major qualitative results of the present study.

There are, of course, serious analytical difficulties associated with a three-dimensional unsteady viscous flow, so that sweeping simplifications are needed to formulate equations that are amenable to analysis. The motion is driven by gravity (surface tension, although undoubtedly important in many real situations, is neglected), and this acts in two ways. Inclination of the bed tends to drive the fluid down the surface in the direction of the fall line, while variations in the depth tend to drive the fluid from deep to shallow regions. It follows that if the slope of the bed is small and the liquid is shallow (i.e. the slope of the free surface is small) then the driving mechanisms are weak, the velocity will be small, and an appropriate Reynolds number can be made small enough to justify the neglect of inertia terms.

The small-slope assumptions have additional ramifications. The velocity in the vertical direction (which is almost perpendicular to the bed) will be small, so that vertical hydrostatic equilibrium prevails with (figure 1)

$$p = -\rho g(z - Q - h), \tag{1.1}$$

where Q is the height of the bed and h is the depth of the fluid. Moreover, vertical velocity gradients will be much larger than horizontal ones, so that the motion in the horizontal plane is governed by

$$0 = -\rho^{-1}\nabla p + \nu\, \partial^2 \mathbf{q}/\partial z^2, \tag{1.2}$$

where $\mathbf{q}$ and the gradient operator have only x and y components. Thus

$$\frac{\partial^2 \mathbf{q}}{\partial z^2} = \frac{g}{\nu}\nabla(h+Q) = \frac{g}{\nu}\nabla h - \frac{\mathbf{F}}{\nu}, \tag{1.3}$$

where $\mathbf{F}$ is the component of the gravity vector tangential to the bed. The velocity $\mathbf{q}$ vanishes at the bed and its first derivative $\partial \mathbf{q}/\partial z$ vanishes at the free surface in the absence of tractions, so that on integrating (1.3) we have

$$\mathbf{q} = \left(\frac{g}{\nu}\nabla h - \frac{\mathbf{F}}{\nu}\right)(\tfrac{1}{2}z^2 - Qz - hz + \tfrac{1}{2}Q^2 + hQ),$$

whence

$$\int_Q^{Q+h} \mathbf{q}\,dz = -\tfrac{1}{3}h^3\left(\frac{g}{\nu}\nabla h - \frac{\mathbf{F}}{\nu}\right), \tag{1.4}$$

which clearly displays the two driving mechanisms alluded to earlier. Since mass conservation requires that

$$\frac{\partial h}{\partial t} + \nabla.\left(\int_Q^{Q+h} \mathbf{q}\,dz\right) = 0$$

we are led to the fundamental equation governing h, namely

$$\frac{g}{12\nu}\Delta(h^4) = \frac{1}{3\nu}\nabla.(h^3\mathbf{F}) + \frac{\partial h}{\partial t}. \tag{1.5}$$

This equation was previously derived by S. H. Smith (1969) and P. Smith (1969), who discuss the underlying approximations a little more carefully.

In general, we are concerned with the solution of (1.5) when there are impenetrable vertical surfaces placed on the bed. The normal velocity vanishes at such a surface, whence

$$g\,\partial h/\partial n = \mathbf{F}.\mathbf{n}, \tag{1.6}$$

where $\mathbf{n}$ is the unit normal. Since there is a boundary layer of thickness $O(h)$ at the surface in which horizontal derivatives must be reinstated, the condition (1.6) is tantamount to assuming that this boundary layer cannot accommodate a significant flux of fluid. Although we offer no analysis of the layer to justify this claim, it should be noted that the only viscous boundary layers whose fundamental role is *other* than merely to adjust the tangential velocity to zero occur in rapidly rotating flows, magnetohydrodynamics and similar situations for which there are large non-conservative body forces. Boundary layers in Hele–Shaw flows are perhaps most similar to the present ones, and they are passive (Thompson 1968). For these reasons the condition (1.6) is a plausible one.

At the leading edge the depth vanishes and we shall assume that it is correct to impose this condition on the solutions of (1.5), i.e.

$$h \to 0 \quad \text{as} \quad x \to x_L(y,t). \tag{1.7}$$

The leading edge is undoubtedly a region of non-uniformity and (1.7) can be thought of as a 'minimum singularity' assumption. It is a fairly reliable rule of asymptotics that if the outer solution can be made to satisfy a boundary condition for the inner solution then it should be imposed. The adjustment of the flow structure required of the inner solution is then minimized and does not have to be considered in deducing the outer solution to first order. Intimately related to this is the assumption that the region of non-uniformity is thin. It will shortly emerge that the horizontal scale of all the problems considered in this paper is at least $O(h/\alpha)$, where α is the bed slope, and the region of non-uniformity must be much smaller than this. It is plausible to conjecture that the various flow parameters can always be chosen to satisfy this requirement. For example, if surface tension must be reinstated close to the leading edge, the size of the region in which this is necessary can be made small by making the surface tension small. If the slope of the free surface is large in a region characterized only by h, this is small compared with h/α provided that α is small. It is because our description is not valid too close to the leading edge that the difficulty discussed by Dussan V. & Davis (1974) concerning the very notion of a fluid continuum within the neighbourhood of a moving contact line plays no role.

A mean velocity $\bar{\mathbf{q}}$ may be defined by

$$\bar{\mathbf{q}} = \frac{1}{h}\int_{Q}^{Q+h} \mathbf{q}\, dz = -\frac{g}{9\nu}\nabla(h^3) + \frac{h^2\mathbf{F}}{3\nu}. \tag{1.8}$$

Approaching any line along which the depth vanishes there are two possibilities. Either $\bar{\mathbf{q}} \to 0$, or $\nabla h \to \infty$ in such a manner that $\bar{\mathbf{q}}$ is not zero, so that

$$\bar{\mathbf{q}} \to (-g/9\nu)\nabla(h^3). \tag{1.9}$$

The second possibility corresponds to a moving leading edge and it is apparent that the mean velocity is perpendicular to the leading edge. It may therefore be anticipated that (1.9) is the leading-edge velocity. Of course, $\bar{\mathbf{q}}$ is parallel to the surface of any obstacle placed on the bed because of the condition (1.6), so that in this way we deduce the important result that a *moving leading edge intersects a solid boundary at right angles.* In §5 this result emerges as a solvability condition.

Equation (1.9) implies that close to the leading edge h behaves like the third power of distance from the leading edge. This non-uniformity does not appear to be of consequence, particularly since the associated shear stress is integrable.

The one-dimensional progressive wave. One-dimensional flow down a plane inclined at an angle α to the horizontal is governed by the equation

$$\frac{g}{12\nu}\frac{\partial^2}{\partial x^2}h^4 = \frac{\alpha g}{3\nu}\frac{\partial}{\partial x}h^3 + \frac{\partial h}{\partial t}. \tag{1.10}$$

It is well known that there are solutions of this equation corresponding to progressive waves of permanent form, and for such solutions

$$h = h(s), \quad s = x - Ut,$$

where U is the wave speed. Of particular significance in so far as the present theory is concerned are solutions corresponding to the advance of fluid over a dry bed. Indeed, integrating (1.10) yields

$$\frac{g}{3\nu}h^2\frac{\partial h}{\partial s} = \frac{\alpha g}{3\nu}h^2 - U, \tag{1.11}$$

and this describes the transition from $h = 0$ at $s = 0$ to a depth $h = h_m$ as $s \to -\infty$ provided that

$$U = g\alpha h_m^2/3\nu. \tag{1.12}$$

This simple problem defines certain natural scales since the fluid depth is $O(h_m)$, the length of the wave is $O(h_m/\alpha)$ and the time it takes to pass a fixed point is $O(\nu/g\alpha^2 h_m)$. These scales reveal that the terms neglected in deducing (1.5) are indeed small provided that

$$\alpha \ll 1, \quad g\alpha^2 h_m^3/\nu^2 \ll 1,$$

and these inequalities ensure the validity of (1.5) for all the problems discussed in this paper. The solution is characterized by a straight leading edge perpendicular to the fall line, and we are concerned with perturbations of this solution (not necessarily small) generated by a variety of disturbances such as unevenness in the bed surface. A number of special problems will be investigated in order to uncover certain general principles.

In order to obtain analytical results a common asymptotic limit is considered. For each problem a natural length L is defined by the geometry (e.g. the channel width) and it is assumed that this is much larger than the length h_m/α associated with the one-dimensional wave. Thus the fundamental small parameter is

$$\epsilon \equiv h_m/\alpha L \ll 1, \tag{1.13}$$

and on the geometrical scale the one-dimensional wave described by (1.11) looks like a semi-infinite slab of fluid of uniform thickness h_m. Distortions of the leading edge are described on a scale characterized by L, so that the structure close to the leading edge is quasi-one-dimensional. Far behind the leading edge, where the structure is two-dimensional, h is governed by a linear equation for each of the problems considered, a consequence of certain restrictions on the magnitude of the disturbances. Thus the equations for both the near field and the far field can be analysed in a straightforward fashion, and when matched essentially complete the analysis.

2. Flow in a rectangular channel

Consider an open channel formed when two straight parallel walls a distance L apart are placed perpendicular to an inclined plane. When the channel axis is parallel to the fall line, the one-dimensional progressive-wave solution satisfies the governing equation (1.5) and the boundary condition (1.6) at each of the walls, and so describes a channel flow. If the channel axis is inclined to the fall line a progressive-wave solution still exists, but it is two-dimensional and the leading edge is deformed. The basic goal of our analysis is to calculate this deformation. The solution, being steady, is somewhat simpler than those discussed in the next three sections and so provides an appropriate introduction to the necessary techniques.

If x is measured in the direction of the channel axis (figure 2), the gravity force can be written as

$$\mathbf{F} = (\alpha, \beta) g, \tag{2.1}$$

where β is a measure of the axis inclination. The boundary condition (1.6) is then

$$\partial h/\partial y = \beta \quad \text{at} \quad y = 0, L, \tag{2.2}$$

and the condition that the leading edge intersects a wall at right angles is

$$\partial x_L/\partial y = 0 \quad \text{at} \quad y = 0, L, \tag{2.3}$$

where the leading edge is described by

$$x = x_L(y, t). \tag{2.4}$$

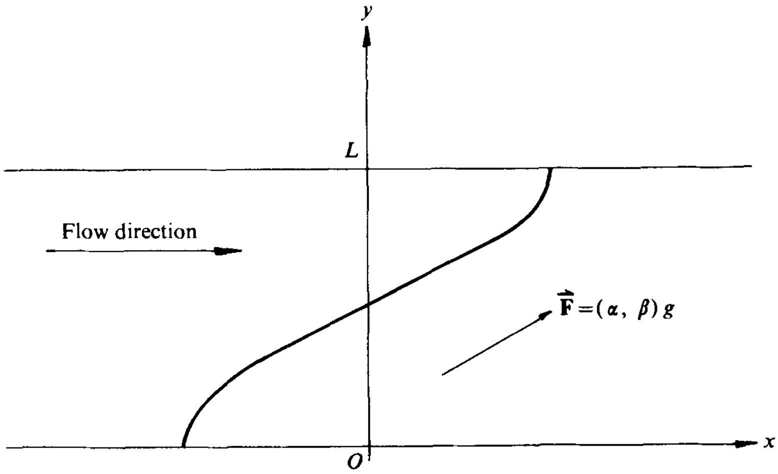

FIGURE 2. Progressive wave in a rectangular channel.

In the physical plane the domain occupied by the fluid changes with time, which is mathematically inconvenient. Consequently we replace x by the new variable

$$s \equiv x - x_L(y,t) \tag{2.5}$$

so that the fluid occupies the region $s < 0$. Then, defining non-dimensional quantities by

$$\left.\begin{aligned} h = h_m\bar{h}, \quad s = \frac{h_m}{\alpha}\bar{s}, \quad y = \frac{h_m}{\alpha}\bar{y}, \quad x_L = \frac{h_m}{\alpha}\bar{x}_L, \quad t = \frac{\nu}{g\alpha^2 h_m}\bar{t}, \\ \mathbf{F} = \alpha g\bar{\mathbf{F}} = \alpha g(\bar{F}_1, \bar{F}_2), \end{aligned}\right\} \tag{2.6}$$

the governing equation becomes, *in general*,

$$\frac{1}{12}\frac{\partial^2}{\partial\bar{s}^2}\bar{h}^4 + \frac{1}{12}\left(\frac{\partial}{\partial\bar{y}} - \frac{\partial\bar{x}_L}{\partial\bar{y}}\frac{\partial}{\partial\bar{s}}\right)^2\bar{h}^4 = \frac{1}{3}\frac{\partial}{\partial\bar{s}}(\bar{h}^3\bar{F}_1) + \frac{1}{3}\left(\frac{\partial}{\partial\bar{y}} - \frac{\partial\bar{x}_L}{\partial\bar{y}}\frac{\partial}{\partial\bar{s}}\right)(\bar{h}^3\bar{F}_2) + \frac{\partial\bar{h}}{\partial\bar{t}} - \frac{\partial\bar{x}_L}{\partial\bar{t}}\frac{\partial\bar{h}}{\partial\bar{s}}, \tag{2.7}$$

where $\bar{F}_1 = 1$ and $\bar{F}_2 = \beta/\alpha$ for the present problem. The motivation for the scalings defined by (2.6) is provided by the one-dimensional progressive wave and it should be noted that h_m is a depth characteristic of the non-uniform flow far behind the leading edge. It will be defined precisely below.

A progressive-wave solution is one for which

$$\bar{x}_L = \bar{U}\bar{t} + \bar{\phi}(\bar{y}), \quad \bar{h} = \bar{h}(\bar{s}, \bar{y}), \tag{2.8a, b}$$

and our fundamental concern is calculation of the wave speed $\bar{U}$ and the shape function $\bar{\phi}$, assuming

$$\epsilon \equiv h_m/\alpha L \ll 1.$$

When β/α vanishes, $\bar{U}$ is equal to $\frac{1}{3}$ [cf. (1.12)] and $\bar{\phi}$ is a constant, corresponding to a straight leading edge perpendicular to the fall line. For sufficiently small values of β/α the solution will be a small perturbation of this one-dimensional wave and it can be anticipated that this perturbation increases in magnitude as β/α is increased until, for suitably large values, the leading edge has an $O(1)$ slope, i.e. $\bar{\phi} = O(1/\epsilon)$. This occurs, in fact, when $\beta/\alpha = O(\epsilon^2)$, so that it is appropriate to consider the limit

$$\beta/\alpha \to 0, \quad \epsilon \to 0, \quad \beta/\alpha = k\epsilon^2, \quad k = O(1) \tag{2.9}$$

and seek solutions corresponding to $O(1)$ deformations. Such a limit is suggested by a small perturbation analysis valid when $\beta/\alpha \to 0$, the details of which are not described herein, and is justified by the ensuing analysis.

It is appropriate to define a scaled distance

$$Y = \epsilon \bar{y}, \tag{2.10}$$

so that the duct walls are located at $Y = 0, 1$, and seek a near-field solution valid when $\bar{s} = O(1)$ of the form

$$\bar{h} \sim h_0(\bar{s}, Y) + \epsilon h_1(\bar{s}, Y) + O(\epsilon^2), \tag{2.11a}$$

$$\bar{U} \sim U_0 + \epsilon U_1 + O(\epsilon^2), \tag{2.11b}$$

$$\bar{\phi} \sim \epsilon^{-1}\chi_0(Y) + \chi_1(Y) + O(\epsilon). \tag{2.11c}$$

Note that $\bar{\phi}$ is large, so that $\partial x_L/\partial y$ is an $O(1)$ quantity.

The far field is described on the scale $S = O(1)$, where

$$S = \epsilon \bar{s}, \tag{2.12}$$

and has the form
$$\bar{h} \sim H_0(S, Y) + \epsilon H_1(S, Y) + O(\epsilon^2). \tag{2.13}$$

The formulation is completed by noting that the boundary conditions at the walls can be written in the form

$$d\bar{\phi}/dY = 0, \quad \partial \bar{h}/\partial Y = k\epsilon \quad \text{at} \quad Y = 0, 1. \tag{2.14a, b}$$

The problem for H_0 is

$$0 = \frac{1}{3}\frac{\partial}{\partial S}(H_0^3) - U_0 \frac{\partial H_0}{\partial S}, \tag{2.15}$$

with solution
$$H_0 = 1. \tag{2.16}$$

On the other hand the problem for h_0, after a single integration, becomes

$$\left.\begin{aligned} \tfrac{1}{12}(1+\chi_0'^2)\,\partial h_0^4/\partial \bar{s} &= \tfrac{1}{3}h_0^3 - U_0 h_0, \\ h_0(0, Y) &= 0, \end{aligned}\right\} \tag{2.17}$$

and this has a solution that matches with (2.16) only if

$$U_0 = \tfrac{1}{3}, \tag{2.18}$$

namely
$$(1+\chi_0'^2)\int_0^{h_0} \frac{t^2\,dt}{(t^2-1)} = \bar{s}. \tag{2.19}$$

The quasi-one-dimensional structure of the near field implies that it cannot in general satisfy the conditions (2.14); there are regions of non-uniformity where $Y = O(\epsilon)$ and where $1 - Y = O(\epsilon)$. Nevertheless the solution (2.19) is compatible with the conditions

$$\chi_0' = 0, \quad \partial h_0/\partial Y = 0 \quad \text{at} \quad Y = 0, 1.$$

The perturbation depth in the far field satisfies

$$\partial H_1/\partial S = 0, \tag{2.20}$$

with solution
$$H_1 = kY. \tag{2.21}$$

Considering (2.16) and (2.21), it is therefore apparent that h_m is the depth far behind the leading edge at the side wall $y = 0$.

Continuing, the problem for h_1 can be written as

$$\tfrac{1}{12}(1+\chi_0'^2)\frac{\partial}{\partial \bar{s}}(4h_0^3h_1) - h_0^2h_1 + \tfrac{1}{3}h_1 = \tfrac{1}{6}\chi_0'\frac{\partial}{\partial Y}h_0^4 - \tfrac{1}{6}\chi_0'\chi_1'\frac{\partial}{\partial \bar{s}}h_0^4 + \tfrac{1}{12}\chi_0''h_0^4 - U_1h_0, \quad (2.22a)$$

$$h_1(0, Y) = 0, \quad \lim_{\bar{s}\to -\infty} h_1(\bar{s}, Y) = kY, \quad (2.22b, c)$$

and it is this that determines the shape function χ_0 and the perturbation wave speed U_1. For (2.22a) can be integrated by writing it in the form

$$\frac{d}{dh_0}\left[\frac{h_0^2h_1}{1-h_0^2}\right] = -\frac{12h_0^4}{(1-h_0^2)^2}\left[\tfrac{1}{6}\chi_0'\frac{\partial h_0}{\partial Y} - \tfrac{1}{6}\chi_0'\chi_1'\frac{\partial h_0}{\partial \bar{s}} + \frac{\chi_0''}{48}h_0 - \frac{U_1}{4h_0^2}\right]$$

and the condition (2.22c) can be satisfied only if

$$\chi_0'' = 12U_1 - 8kY. \quad (2.23)$$

Since χ_0' vanishes at each wall, it follows that

$$U_1 = \tfrac{1}{3}k \quad (2.24a)$$

and $$\chi_0 = -\tfrac{4}{3}k(Y-\tfrac{1}{2})^3 + k(Y-\tfrac{1}{2}) + \text{constant}. \quad (2.24b)$$

The profile (2.24b) is sketched in figure 2.

The most striking feature of this solution is the extreme sensitivity to the data that it displays. When $h_m/\alpha L$ is very small, only a tiny $[O(h_m/\alpha L)^2]$ inclination of the channel axis to the fall line is needed to generate significant leading-edge deformations. This pathological behaviour has its counterpart in each of the problems that we shall consider.

3. Flow over an uneven bed

Steady solutions of (1.5) corresponding to flow over an uneven bed have been considered by P. Smith (1969) and by S.H. Smith (1969). Here the concern is with an unsteady flow in which a one-dimensional progressive wave advancing over an inclined plane penetrates a region where the bed is no longer flat, thus causing the leading edge to deviate from a straight line. The problem is to describe the history of this deviation. The flow is genuinely unsteady, unlike that of § 2, but the analysis is pursued in a very similar fashion.

The flow is governed by (2.7) with the gravity force given by

$$\left.\begin{aligned}\bar{\mathbf{F}} &= (1,0) + \delta f(X,Y) \equiv (1,0) + \delta(f_1, f_2),\\ X &= \epsilon\bar{x}, \quad Y = \epsilon\bar{y}.\end{aligned}\right\} \quad (3.1)$$

L is here a length characteristic of the scale on which the bed deviates from a flat surface and δ is a measure of the magnitude of this deviation. h_m is the uniform depth far behind the leading edge, where f vanishes and the bed is a plane inclined to the horizontal at an angle α.

It is appropriate to define a slow time $T = \epsilon\bar{t}$ and then seek a solution for which the position of the leading edge is described in the slow variables, i.e.

$$x_L \sim \epsilon^{-1}\chi_0(Y,T) + O(1). \quad (3.2)$$

In the vicinity of the leading edge ($\bar{s} = O(1)$) the argument of $f(X, Y)$ is simplified to

$$f \sim f(\chi_0(Y, T), Y) \tag{3.3}$$

and writing $$h \sim h_0(\bar{s}, Y, T) + O(\epsilon)$$

gives a description which is quasi-one-dimensional:

$$\frac{1}{12}\left[1+\left(\frac{\partial \chi_0}{\partial Y}\right)^2\right]\frac{\partial^2}{\partial \bar{s}^2}h_0^4 = \frac{1}{3}\frac{\partial}{\partial \bar{s}}[h_0^3\hat{F}_1] - \frac{1}{3}\frac{\partial \chi_0}{\partial Y}\frac{\partial}{\partial \bar{s}}[h_0^3\hat{F}_2] - \frac{\partial \chi_0}{\partial T}\frac{\partial h_0}{\partial \bar{s}}, \tag{3.4}$$

where $$\hat{F}_1 = 1 + \delta f_1(\chi_0, Y), \quad \hat{F}_2 = \delta f_2(\chi_0, Y).$$

This describes a wave with a zero depth at $\bar{s} = 0$ and a depth

$$H_\infty = \left[\frac{3\partial\chi_0/\partial T}{\hat{F}_1 - (\partial\chi_0/\partial Y)\hat{F}_2}\right]^{\frac{1}{3}} \tag{3.5}$$

as $\bar{s} \to -\infty$.

Far from the leading edge ($\bar{s} = O(1/\epsilon)$) the solution has the form

$$\bar{h} \sim H_0(S, Y, T) + O(\epsilon),$$

where $S = \epsilon\bar{s}$, and essentially the full argument of f must be retained, i.e.

$$f \sim f(S + \chi_0, Y).$$

H_0 then satisfies the equation

$$0 = \frac{1}{3}\frac{\partial}{\partial S}[H_0^3(1+\delta f_1)] + \frac{1}{3}\left(\frac{\partial}{\partial Y} - \frac{\partial \chi_0}{\partial Y}\frac{\partial}{\partial S}\right)[H_0^3\delta f_2] + \frac{\partial H_0}{\partial T} - \frac{\partial \chi_0}{\partial T}\frac{\partial H_0}{\partial S}. \tag{3.6}$$

If for all negative times the fluid is confined to the flat portion of the bed (where $f = 0$), it is appropriate to choose $H_0 = 1$ at $T = 0$. Equation (3.6) then describes how H_0 changes along certain world lines, and indeed if χ_0 is known, H_0 is known uniquely at any point through which only one world line passes, provided this line originates at $T = 0$. In particular, H_0 is known on any world line originating at $T = 0$ that reaches the plane $S = 0$, where H_0 is known *independently* to have the value H_∞ given in (3.5) in order to match with the leading-edge solution. The two values will be equal only if χ_0 is appropriately chosen.

The procedure implied by these remarks can easily be carried out when the deformation of the bed is small, i.e. $\delta \ll 1$. The solution is then a small perturbation of the one-dimensional wave, so that

$$\chi_0 \sim \tfrac{1}{3}T + \delta\phi(Y, T)$$

and $$H_\infty \sim 1 + \delta[\tfrac{3}{2}\partial\phi/\partial T - \tfrac{1}{2}f_1(\tfrac{1}{3}T, Y)] + O(\delta^2). \tag{3.7}$$

Moreover, $$H_0 \sim 1 + \delta J_1 + O(\delta^2),$$

where $$0 = \frac{\partial J_1}{\partial T} + \frac{2}{3}\frac{\partial J_1}{\partial S} + \frac{1}{3}\left[\frac{\partial f_1}{\partial S}(S + \tfrac{1}{3}T, Y) + \frac{\partial f_2}{\partial Y}(S + \tfrac{1}{3}T, Y)\right]. \tag{3.8}$$

Since J_1 vanishes at $T = 0$, the solution of (3.8) on the characteristic originating at $S = S_0$, $T = 0$ is

$$J_1 = -\frac{1}{3}\int_0^T dT'\left[\frac{\partial f_1}{\partial T'}(S_0 + T', Y) + \frac{\partial f_2}{\partial Y}(S_0 + T', Y)\right].$$

The characteristic crossing $S = 0$ at time T originates at $S_0 = -\frac{2}{3}T$, so that matching with (3.7) implies

$$\frac{\partial \phi}{\partial T} = \tfrac{1}{9}f_1(\tfrac{1}{3}T, Y) + \tfrac{8}{9}f_1(-\tfrac{2}{3}T, Y) - \frac{2}{9}\int_0^T dT' \frac{\partial f_2}{\partial Y}(-\tfrac{2}{3}T + T', Y).$$

The leading edge is initially straight ($\phi = 0$ at $T = 0$), so that upon integration

$$\phi(Y,T) = \frac{1}{3}\int_0^{\frac{1}{3}T} dT' f_1(T', Y) + \frac{2}{9}\int_0^T dT' f_1(-\tfrac{2}{3}T', Y)$$
$$-\tfrac{2}{9}T\int_{-\frac{2}{3}T}^{\frac{1}{3}T} dT' \frac{\partial f_2}{\partial Y}(T', Y) + 2\int_0^{\frac{1}{3}T} dT'\, T' \frac{\partial f_2}{\partial Y}(T', Y).$$

If at $\bar{t} = 0$ the straight leading edge is located at $\bar{x} = 0$ it follows, by postulate, that both f_1 and f_2 are identically zero when their first argument is negative. Therefore

$$\phi(Y,T) = \int_0^{\frac{1}{3}T} dT' \left[\tfrac{1}{3}f_1(T', Y) + (2T' - \tfrac{2}{9}T)\frac{\partial f_2}{\partial Y}(T', Y)\right], \tag{3.9}$$

the central result of this section.

It is interesting to consider the effect of an isolated bump or mound upon the leading edge. Consider, for example, a mound whose elevation relative to the inclined plane is characterized (when δ is positive) by

$$\Phi = \begin{cases}(1-R^2)^2, & R \leqslant 1,\\ 0, & R > 1,\end{cases} \quad R^2 = (X-1)^2 + Y^2, \tag{3.10}$$

in the sense that $f = -\nabla\Phi(X, Y)$. The bed slope defined by (3.10) is continuous and it should be noted that an equivalent hollow or depression is obtained by changing the sign of Φ, which merely reverses the sign of ϕ. Equation (3.9) then provides the following solution for ϕ:

$$\phi = 0 \quad \text{for} \quad |Y| > 1, \tag{3.11a}$$

$$\phi = 0 \quad \text{for} \quad |Y| < 1, \quad \tfrac{1}{3}T < 1-(1-Y^2)^{\frac{1}{2}} \tag{3.11b}$$

$$\phi = -\tfrac{1}{3}\Phi(\tfrac{1}{3}T, Y) + 8\int_{1-(1-Y^2)^{\frac{1}{2}}}^{\frac{1}{3}T} dt\,(t - \tfrac{1}{9}T)\,[1-(t-1)^2 - 3Y^2]$$
$$\text{for} \quad |Y| < 1, \quad 1-(1-Y^2)^{\frac{1}{2}} < \tfrac{1}{3}T < 1+(1-Y^2)^{\frac{1}{2}}; \tag{3.11c}$$

$$\phi = \tfrac{32}{3}(1 - \tfrac{1}{9}T)(1-Y^2)^{\frac{3}{2}}(1-4Y^2) \quad \text{for} \quad |Y| < 1, \quad \tfrac{1}{3}T > 1+(1-Y^2). \tag{3.11d}$$

Profiles for several values of T are sketched in figure 3.

It is noteworthy that the mound generates a 'permanent' deformation of the leading edge, namely (3.11d). This grows with time, so that the result cannot be uniformly valid and on a large enough time scale the deformation will be much larger than $O(\delta)$. Deviations from (3.11d) can in fact be expected for times $T = O(1/\epsilon)$ since (3.4) is obtained by neglecting, amongst other terms,

$$\frac{\epsilon}{12}\frac{\partial^2 \chi_0}{\partial Y^2}\frac{\partial}{\partial \bar{s}}h_0^4,$$

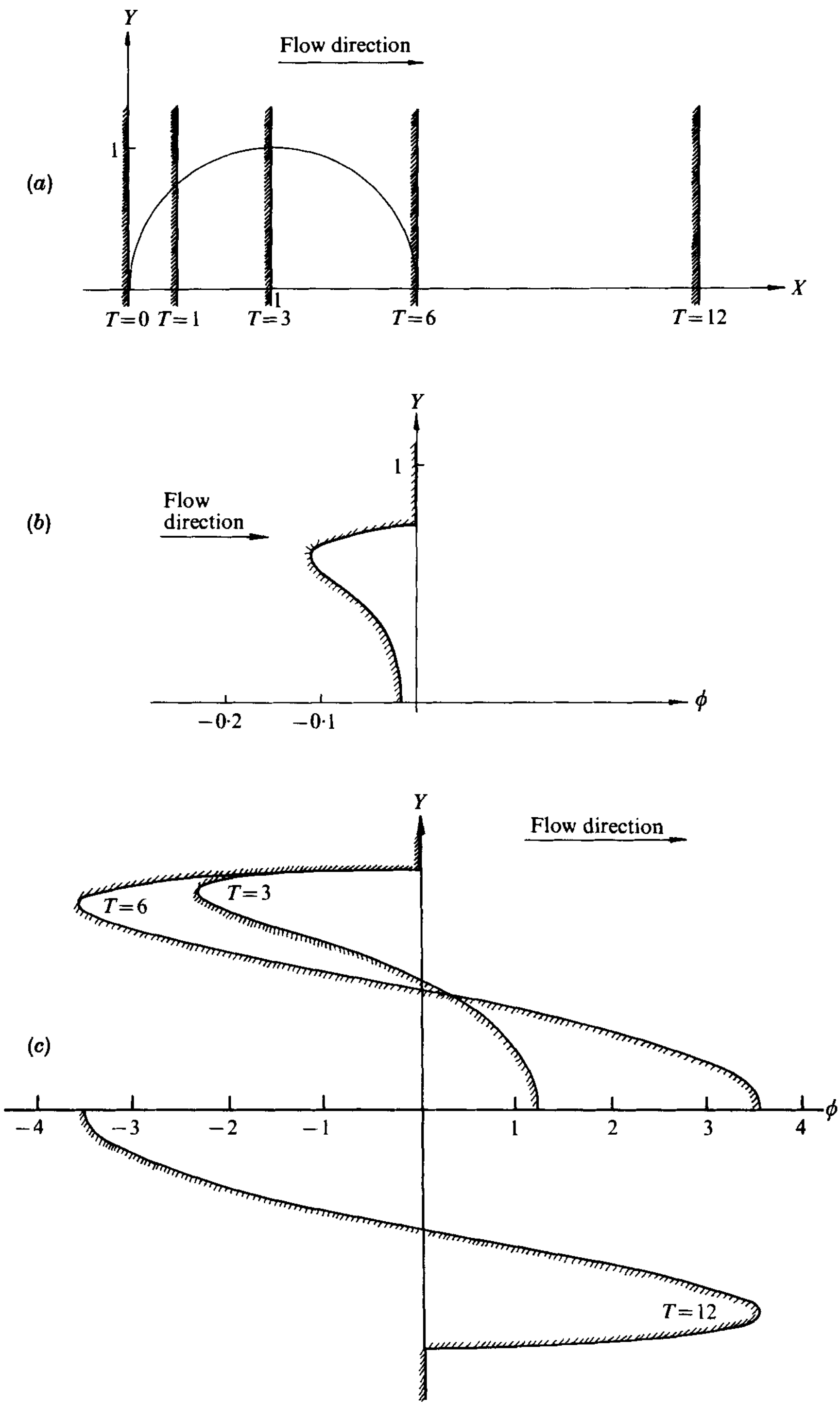

FIGURE 3. Flow over a bump, $T = O(1)$. (*a*) Location of leading edge at different times, neglecting the deformation. (*b*) Deformed leading edge at $T = 1$. (*c*) Deformation at $T = 3$, 6 and 12. At $T = 0$ and 9 the leading edge is straight. Note that only the half of the profile is sketched for each time.

and this will play a role on the much longer time scale. The eventual magnitude of the deformation depends upon δ and when this is $O(\epsilon)$ the leading edge has an $O(1)$ slope during part of its motion, as the following analysis reveals.

We write

$$\overline{\mathbf{F}} = (1,0)+\epsilon k f(X, Y), \quad k = O(1), \tag{3.12}$$

which is (3.1) with δ replaced by ϵk. The leading-edge deformation created by such a disturbance is then appropriately described in terms of multiple time scales and has the form

$$\bar{x}_L \sim \epsilon^{-1}\chi_0(Y,T,T_1)+\chi_1(Y,T,T_1)+o(1), \tag{3.13}$$

where

$$T_1 = \epsilon T = \epsilon^2 \bar{t}.$$

The corresponding near-field structure is

$$\bar{h} \sim h_0(\bar{s}, Y,T,T_1)+\epsilon h_1(\bar{s}, Y,T,T_1)+O(\epsilon^2) \tag{3.14}$$

and in the far field

$$\bar{h} \sim H_0(S, Y,T,T_1)+\epsilon H_1(S, Y,T,T_1)+O(\epsilon^2). \tag{3.15}$$

The function h_0 satisfies the equation

$$\frac{1}{12}\left[1+\left(\frac{\partial\chi_0}{\partial Y}\right)^2\right]\frac{\partial}{\partial\bar{s}}(h_0^4) = \tfrac{1}{3}h_0^3-\frac{\partial\chi_0}{\partial T}h_0, \tag{3.16}$$

so that

$$h_0 \to (3\partial\chi_0/\partial T)^{\frac{1}{2}} \quad \text{as} \quad \bar{s}\to-\infty,$$

where it must match with the solution of

$$\left(H_0^2-\frac{\partial\chi_0}{\partial T}\right)\frac{\partial H_0}{\partial S}+\frac{\partial H_0}{\partial T} = 0, \tag{3.17}$$

namely $H_0 = 1$. (As before, it is assumed that at $T = 0$ the flow is simply the one-dimensional wave.) Thus

$$\chi_0 = \tfrac{1}{3}T+\sigma(Y,T_1), \tag{3.18}$$

where σ is at present unknown.

Turning to perturbation quantities, h_1 satisfies the equation

$$\begin{aligned}\frac{1}{12}\left[1+\left(\frac{\partial\chi_0}{\partial Y}\right)^2\right]&\frac{\partial}{\partial\bar{s}}(4h_0^3h_1)-h_0^2h_1+\frac{\partial\chi_0}{\partial T}h_1\\ &= \frac{1}{12}\left[\frac{\partial\chi_0}{\partial Y}\left(\frac{\partial}{\partial Y}-\frac{\partial\chi_1}{\partial Y}\frac{\partial}{\partial\bar{s}}\right)+\frac{\partial\chi_0}{\partial Y}\frac{\partial}{\partial Y}-\frac{\partial\chi_1}{\partial Y}\frac{\partial\chi_0}{\partial Y}\frac{\partial}{\partial\bar{s}}+\frac{\partial^2\chi_0}{\partial Y^2}\right]h_0^4\\ &\quad-\left(\frac{\partial\chi_0}{\partial T_1}+\frac{\partial\chi_1}{\partial T}\right)h_0+\frac{k}{3}h_0^3f_1(\chi_0, Y)-\frac{k}{3}h_0^3\frac{\partial\chi_0}{\partial Y}f_2(\chi_0, Y),\end{aligned} \tag{3.19}$$

so that as $\bar{s}\to-\infty$

$$h_1\to -\frac{1}{8}\frac{\partial^2\sigma}{\partial Y^2}+\frac{3}{2}\left(\frac{\partial\sigma}{\partial T_1}+\frac{\partial\chi_1}{\partial T}\right)-\tfrac{1}{2}kf_1(\chi_0, Y)+\tfrac{1}{2}k\frac{\partial\sigma}{\partial Y}f_2(\chi_0, Y). \tag{3.20}$$

On the other hand, in the far field

$$\frac{\partial H_1}{\partial T}+\frac{2}{3}\frac{\partial H_1}{\partial S} = -\frac{k}{3}\frac{\partial f_1}{\partial S}(\chi_0+S, Y)-\frac{k}{3}\left[\frac{\partial}{\partial Y}-\frac{\partial\sigma}{\partial Y}\frac{\partial}{\partial S}\right]f_2(\chi_0+S, Y), \tag{3.21}$$

which has its counterpart in (3.8). This equation may be integrated along characteristics originating at $T = 0$, where H_1 vanishes, so that on denoting the right side of (3.21) by $G(S, T)$ (the dependence on Y and T_1 is then implicit) it is easily shown that as $S \to 0$

$$H_1 \to \int_{-\frac{2}{3}T}^{0} G(S', \tfrac{3}{2}S' + T)\, dS', \tag{3.22}$$

which must agree with (3.20). This can be thought of as an equation for $\partial\chi_1/\partial T$.

Now the key to choosing σ is the requirement that (3.13) be uniformly valid and the simplest way of guaranteeing this is to impose the condition

$$\lim_{T\to\infty} (\partial\chi_1/\partial T) = 0. \tag{3.23}$$

For a mound of finite dimensions, the function $f(\chi_0, Y)$ also vanishes in this limit, so that in this way the basic equation governing σ is uncovered, namely

$$\frac{\partial\sigma}{\partial T_1} - \frac{1}{12}\frac{\partial^2\sigma}{\partial Y^2} = \lim_{T\to\infty} \frac{2}{3}\int_{-\frac{2}{3}T}^{0} G(S', \tfrac{3}{2}S' + T)\, dS'. \tag{3.24}$$

The right-hand side is
$$-\tfrac{2}{9}k \int_{-\infty}^{\infty} dt \frac{\partial f_2}{\partial Y}(t, Y),$$

so that for the mound described by (3.10)

$$\frac{\partial\sigma}{\partial T_1} - \frac{1}{12}\frac{\partial^2\sigma}{\partial Y^2} = \begin{cases} -\tfrac{32}{27}k(1-4Y^2)(1-Y^2)^{\frac{1}{2}}, & |Y| \leqslant 1, \\ 0, & |Y| > 1, \end{cases} \tag{3.25}$$

which must be solved subject to the condition that σ vanishes when T_1 does, corresponding to an initially straight leading edge.

For small values of T_1

$$\sigma \sim \begin{cases} -\tfrac{32}{27}k(1-4Y^2)(1-Y^2)^{\frac{1}{2}}T_1, \quad |Y| \leqslant 1, & (3.26a) \\ 0, \quad |Y| > 1, & (3.26b) \end{cases}$$

and the time-dependent part of (3.11d) is recovered. The time-independent part is contained in χ_1, which we do not calculate.

The solution of (3.25) is

$$\sigma = -\tfrac{32}{27}k\left(\frac{12T_1}{\pi}\right)^{\frac{1}{2}} \int_{-1}^{+1} d\xi (1-4\xi^2)(1-\xi^2)^{\frac{1}{2}} \left[e^{-\theta} - 2\theta^{\frac{1}{2}} \int_{\theta^{\frac{1}{2}}}^{\infty} dt \exp(-t^2)\right], \tag{3.27}$$
$$\theta = 3(Y-\xi)^2/T_1,$$

and for large values of T_1, with $Y/T_1^{\frac{1}{2}}$ fixed,

$$\sigma \sim \tfrac{24}{27}k(3\pi)^{\frac{1}{2}} \frac{1}{T_1^{\frac{1}{2}}} \exp\left(-\frac{3Y^2}{T_1}\right), \tag{3.28}$$

so that most of the leading edge, after suffering large distortions, eventually straightens out. However, when Y is fixed the large-time behaviour is

$$\sigma \sim \begin{cases} -\dfrac{128k}{45}(1-Y^2)^{\frac{5}{2}}, \quad |Y| \leqslant 1, & (3.29a) \\ 0, \quad |Y| \geqslant 1, & (3.29b) \end{cases}$$

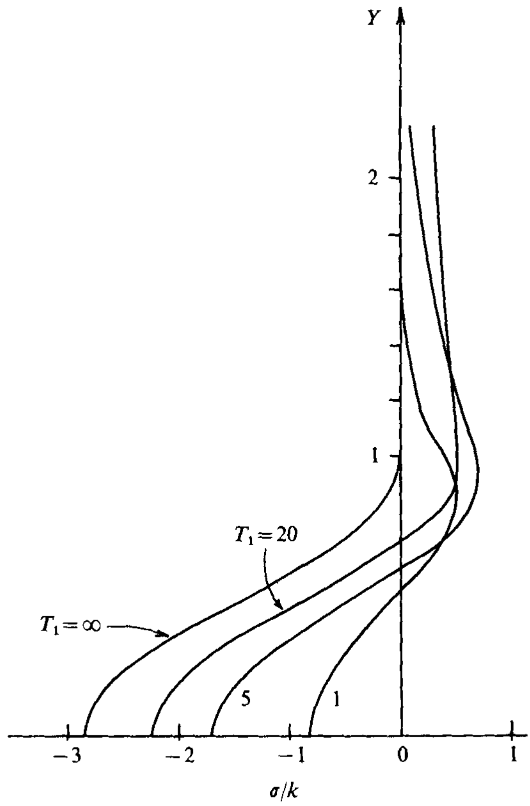

FIGURE 4. Flow over a bump, $T = O(1/\epsilon)$.

so that there is a residual disturbance. Profiles obtained by evaluating (3.27) at different times describe the transition between (3.26) and (3.29) and are shown in figure 4.

In summary, when the leading edge traverses a mound of diameter L and height $O(h_m)$, which it does on the time scale $T = O(1)$, deformations of slope $O(\epsilon)$ are generated. These deformations continue to grow once the mound has been left behind, and on the time scale $T = O(1/\epsilon)$ the leading edge has an $O(1)$ slope which ultimately decays to zero except for a residual deformation in the region $|Y| < 1$.

4. Flow down an open channel of circular cross-section

The problem discussed in this section concerns a flow that is often observed when a viscous liquid is poured from a round-necked bottle. Consider a right-circular pipe inclined at an angle α to the horizontal and down which a small amount of fluid is flowing. Specifically, the maximum depth h_m (figure 5) is assumed to satisfy the inequality

$$h_m \ll \alpha^2 R, \tag{4.1}$$

where R is the radius of the pipe. For such small depths the width L of the fluid is much smaller than R, and elementary geometrical considerations imply that

$$h_m/L = O(L/R), \tag{4.2}$$

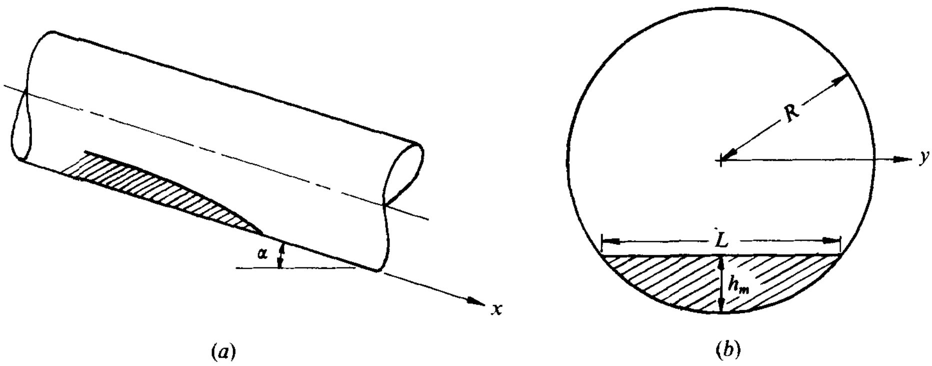

FIGURE 5. Flow in a partially filled pipe. (*a*) Side view. (*b*) Cross-section.

so that because of (4.1) $L \ll \alpha R$ and, furthermore, $\epsilon = h_m/\alpha L \ll 1$. ϵ is the fundamental small parameter of the problem, as always. The force $\mathbf{F}$ is then given approximately by

$$\mathbf{F} \sim (\alpha, -y/R)g. \tag{4.3}$$

There is a steady solution for which h is a function of y only, and for which (1.5) simplifies to

$$\frac{\partial^2}{\partial y^2}h^4 = -\frac{4}{R}\frac{\partial}{\partial y}(h^3 y),$$

with solution

$$h = h_m - y^2/2R \equiv (L^2 - 4y^2)/8R. \tag{4.4}$$

Thus the surface of the fluid is flat. The mean velocity defined by (1.8) has only an x component, namely

$$\bar{q}_1 = \frac{1}{3}\frac{\alpha g}{\nu}h^2 = \frac{1}{3}\frac{\alpha g}{\nu}\left(\frac{1}{8R}\right)^2(L^2 - 4y^2)^2. \tag{4.5}$$

These elementary calculations provide all the information that is needed to calculate the shape of the leading edge. Bear in mind that on the $O(L)$ scale the leading edge is simply a discontinuity in h; moreover the leading edge moves with the limiting mean velocity. Thus an intuitively satisfying picture of the motion is one for which, away from the leading edge, h is described by (4.4), and the leading edge itself is displaced in the x direction at a rate given by (4.5). Mathematically, the leading-edge structure then has two roles: it must smooth out the discontinuity in h and also rotate the vector $\mathbf{q}$ from an x-wise orientation to a direction perpendicular to the leading edge. The formal analysis which confirms this picture proceeds as follows.

We write

$$\bar{x}_L \sim \epsilon^{-1}\chi_0(Y,T) + O(1) \tag{4.6}$$

and note that $\bar{\mathbf{F}}$ has the form

$$\bar{\mathbf{F}} \sim (1,0) - (L/\alpha R)(0, -Y) + O(\epsilon^2), \tag{4.7}$$

where $L/\alpha R$ is an $O(\epsilon)$ quantity. Note that $\partial x_L/\partial y$ is $O(1)$, i.e. the formulation permits the leading edge to have an $O(1)$ slope.

In the neighbourhood of the leading edge

$$\bar{h} \sim h_0(\bar{s}, Y, T) + O(\epsilon),$$

so that

$$\frac{1}{12}\left[1 + \left(\frac{\partial \chi_0}{\partial Y}\right)^2\right]\frac{\partial^2}{\partial \bar{s}^2}h_0^4 = \frac{1}{3}\frac{\partial}{\partial \bar{s}}h_0^3 - \frac{\partial \chi_0}{\partial T}\frac{\partial h_0}{\partial \bar{s}}. \tag{4.8}$$

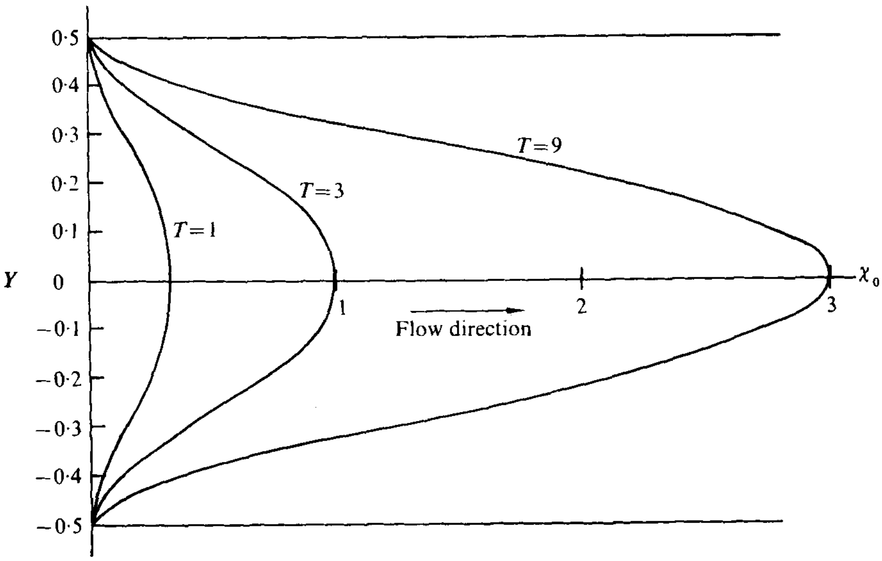

FIGURE 6. Leading-edge deformation for pipe flow.

As with the earlier problems, this is a quasi-one-dimensional wave whose structure is modulated on the slow scales Y and T. For large values of $-\bar{s}$, h_0 approaches an asymptotic limit

$$H_\infty = [3\partial\chi_0/\partial T]^{\frac{1}{2}}. \tag{4.9}$$

The solution far from the leading edge has the form

$$\bar{h} \sim H_0(S, Y, T) + O(\epsilon), \quad S = \epsilon\bar{s},$$

whence
$$0 = \left(H_0^2 - \frac{\partial\chi_0}{\partial T}\right)\frac{\partial H_0}{\partial S} + \frac{\partial H_0}{\partial T}. \tag{4.10}$$

Provided that $H_0^2 > \partial\chi_0/\partial T$, H_0 does not change along characteristics that originate at $T = 0, S < 0$ and intersect $S = 0$ at positive values of T. Thus if H_0 is specified initially it can be calculated for all positive times at $S = 0$, where it must agree with the result (4.9). In this way the velocity of the leading edge can be calculated. In particular, if H_0 is initially described by the steady-state solution (4.4), namely

$$H_0(S, Y, 0) = 1 - 4Y^2,$$

then
$$\partial\chi_0/\partial T = \tfrac{1}{3}(1 - 4Y^2)^2, \tag{4.11}$$

which confirms the earlier conjecture that the leading edge is displaced in the x direction at the speed $\bar{q}_1$ given by (4.5). The time history of an initially straight leading edge is sketched in figure 6.

For more general initial conditions the situation is more complicated: (4.10) might not have a unique solution, for example. It can be plausibly conjectured that the result (4.11) is then significant in an asymptotic sense, for large values of T. For it is surely reasonable to suppose that far behind the leading edge the solution approaches (4.4) at large times whatever the initial data. And this solution is carried throughout the region $S < 0$ by the characteristics defined by (4.10).

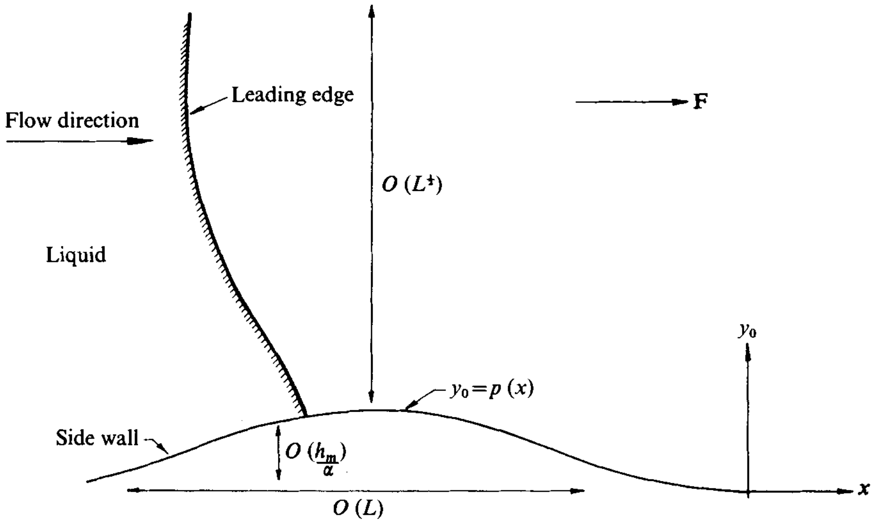

FIGURE 7. Flow past a side wall.

5. Flow past a single side wall

The fourth and final problem to be examined in this paper is flow over an inclined plane upon which is located a single side wall. This wall is not flat and so generates unsteady disturbances as the leading edge moves down the bed.

The co-ordinate system is chosen such that the fall line is aligned with the x axis (figure 7); then $\mathbf{F} = (\alpha g, 0)$. y is defined a little differently from before in that if y_0 is the Cartesian ordinate, then $y = y_0 - p(x)$, where the side wall is located at $y_0 = p(x)$, i.e. $y = 0$. The definition of s is unchanged, i.e. $s = x - x_L(y,t)$. Variables are non-dimensionalized in the familiar way (2.6), and at the same time we write

$$p(x) = (h_m/\alpha)\,P(X), \quad X = \epsilon\bar{x}, \quad \epsilon = h_m/\alpha L, \tag{5.1}$$

so that L is a length which defines the scale on which the wall displacement varies. The equation governing the variation of h in the region $\bar{y} \geqslant 0$, $\bar{s} \leqslant 0$ is then no longer (2.7) but rather

$$\left(\frac{\partial}{\partial\bar{s}} - \epsilon P'\frac{\partial}{\partial\bar{y}} + \epsilon P'\frac{\partial\bar{x}_L}{\partial\bar{y}}\frac{\partial}{\partial\bar{s}}\right)^2\bar{h}^4 + \left(\frac{\partial}{\partial\bar{y}} - \frac{\partial\bar{x}_L}{\partial\bar{y}}\frac{\partial}{\partial\bar{s}}\right)^2\bar{h}^4$$
$$= 4\left(\frac{\partial}{\partial\bar{s}} - \epsilon P'\frac{\partial}{\partial\bar{y}} + \epsilon P'\frac{\partial\bar{x}_L}{\partial\bar{y}}\frac{\partial}{\partial\bar{s}}\right)(\bar{h}^3) + 12\left(\frac{\partial}{\partial\bar{t}} - \frac{\partial\bar{x}_L}{\partial\bar{t}}\frac{\partial}{\partial\bar{s}}\right)\bar{h}. \tag{5.2}$$

The tangency condition (1.6) becomes

$$\epsilon P'\left(\frac{\partial\bar{h}}{\partial\bar{s}} - \epsilon P'\frac{\partial\bar{h}}{\partial\bar{y}} + \epsilon P'\frac{\partial\bar{x}_L}{\partial\bar{y}}\frac{\partial\bar{h}}{\partial\bar{s}} - 1\right) = \frac{\partial\bar{h}}{\partial\bar{y}} - \frac{\partial\bar{x}_L}{\partial\bar{y}}\frac{\partial\bar{h}}{\partial\bar{s}} \quad \text{at} \quad \bar{y} = 0. \tag{5.3}$$

$\partial\bar{h}/\partial\bar{s}$ is unbounded as $\bar{s} \to 0$, of course, so that at first glance this equation is suspect, but the condition that the leading edge intersects the wall at right angles is

$$\epsilon P'(1 + \epsilon P'\partial\bar{x}_L/\partial\bar{y}) = -\partial\bar{x}_L/\partial\bar{y} \quad \text{at} \quad \bar{y} = 0, \quad \bar{s} = 0, \tag{5.4}$$

and there is in fact no difficulty.

It is appropriate to write

$$\bar{x}_L \sim \frac{1}{\epsilon}\frac{T}{3} + \frac{1}{\epsilon^{\frac{1}{2}}}\chi_0(Y,T) + \chi_1(Y,T) + O(\epsilon^{\frac{1}{2}}), \tag{5.5}$$

where
$$T = \epsilon\bar{t}, \quad Y = \epsilon^{\frac{1}{2}}\bar{y},$$

a formulation that permits the leading edge to have an $O(1)$ slope. This is fitting for wall displacements as large as those defined by (5.1). Note that the slow variable Y is defined differently from that of earlier sections.

The solution for $\bar{h}$ must be constructed in two separate regions, as always. Close to the leading edge

$$\bar{h} \sim h_0(\bar{s}, Y, T) + \epsilon^{\frac{1}{2}}h_1(\bar{s}, Y, T) + O(\epsilon),$$

whence
$$\left[1 + \left(\frac{\partial\chi_0}{\partial Y}\right)^2\right]\int_0^{h_0}\frac{t^2\,dt}{(t^2-1)} = \bar{s}. \tag{5.6}$$

This describes the transition from $h_0 = 0$ at $\bar{s} = 0$ to $h_0 \to 1$ as $\bar{s} \to -\infty$ and justifies the choice of the first term in (5.5). Just as for the rectangular-channel problem discussed in §2, the leading-edge solution is not uniformly valid as $Y \to 0$, but nevertheless (5.6) is compatible with the first-order wall conditions

$$\partial\chi_0/\partial Y = 0, \quad \partial h_0/\partial Y = 0 \quad \text{at} \quad Y = 0.$$

The equation for h_1 is

$$\left[1 + \left(\frac{\partial\chi_0}{\partial Y}\right)^2\right]\frac{\partial}{\partial\bar{s}}(4h_0^3h_1) - 12h_0^2h_1 + 4h_1$$
$$= h_0^4\frac{\partial^2\chi_0}{\partial Y^2} - 2\left(\frac{\partial\chi_0}{\partial Y}\right)\left(\frac{\partial\chi_1}{\partial Y}\right)\frac{\partial}{\partial\bar{s}}h_0^4 - 12\frac{\partial\chi_0}{\partial T}h_0 + 2\frac{\partial\chi_0}{\partial Y}\frac{\partial}{\partial Y}h_0^4, \tag{5.7}$$

and as $\bar{s} \to -\infty$
$$h_1 \to H_\infty(Y,T) \equiv \left(-\frac{1}{8}\frac{\partial^2\chi_0}{\partial Y^2} + \frac{3}{2}\frac{\partial\chi_0}{\partial T}\right). \tag{5.8}$$

Far from the leading edge the solution has the form

$$\bar{h} \sim 1 + \epsilon^{\frac{1}{2}}H_1(S, Y, T) + O(\epsilon), \quad S = \epsilon s,$$

where H_1 satisfies the equation
$$\frac{\partial^2 H_1}{\partial Y^2} = 2\frac{\partial H_1}{\partial S} + 3\frac{\partial H_1}{\partial T}. \tag{5.9}$$

H_1 must take the value H_∞ at $S = 0$, and furthermore the tangency condition is

$$\partial H_1/\partial Y = -P'(S + \tfrac{1}{3}T) \quad \text{at} \quad Y = 0. \tag{5.10}$$

Suppose that, for all times T less than some value T_0, that part of the wall that is in contact with the fluid is flat. Then

$$H_1 = 0 \quad \text{at} \quad T = T_0. \tag{5.11}$$

Now there is no question of being able to satisfy the system (5.9)–(5.11) for arbitrary values of H_∞. Loosely speaking (5.9) can be thought of as a heat equation, S being a time-like variable. Since S is negative the specification of initial data at $S = 0$ is therefore analogous to integrating the heat equation backwards in time. This cannot be done for arbitrary boundary and initial data, but for given boundary data there is a

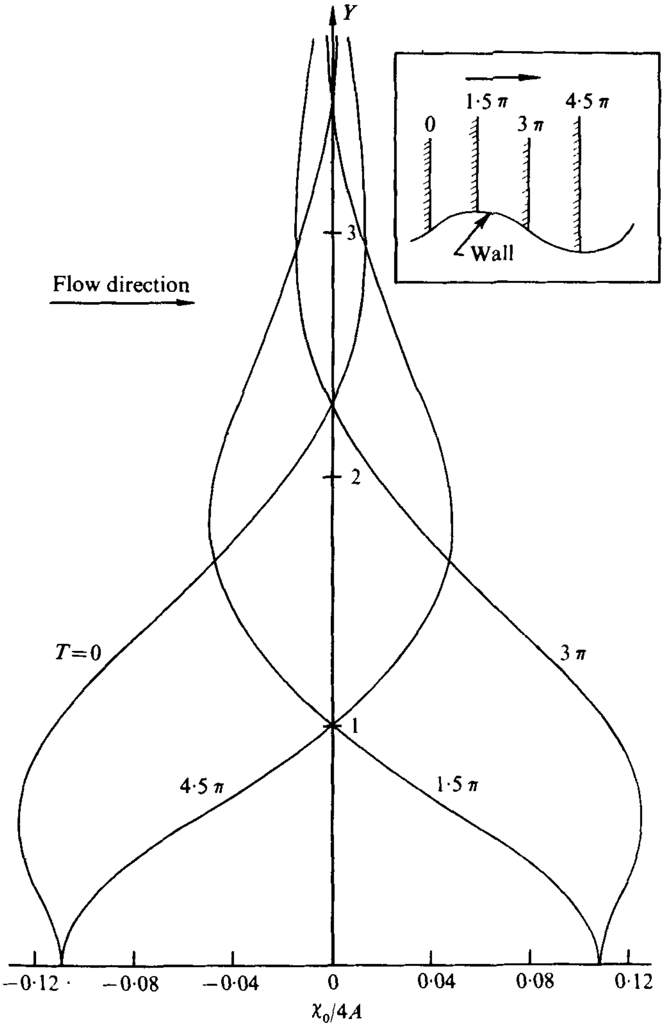

FIGURE 8. Deformations generated by a wavy wall. The inset shows the nominal location of the leading edge at different times, the corresponding deformation being shown in the main part of the figure.

unique set of initial data for which a solution exists. Thus it is not surprising that (5.9), subject to the conditions (5.10) and (5.11) and appropriate boundedness conditions at $Y = \infty$ and $S = -\infty$, has only one solution. Thus H_∞ is uniquely determined once P is specified and this leads to an equation for the shape function χ_0. A simple uniqueness proof can be given (L. E. Payne, private communication; see appendix) if certain assumptions are made about how rapidly the solution dies out at infinity.

The existence of a solution is established by explicit construction. Consider a periodic wall displacement

$$P'(S+\tfrac{1}{3}T) = A\exp\left[i\omega(S+\tfrac{1}{3}T)\right]. \tag{5.12}$$

The solution of (5.9) subject to (5.10) is

$$H_1 = A\exp\left[i\omega(S+\tfrac{1}{3}T)\right]\left(\frac{2}{3|\omega|}\right)^{\frac{1}{3}}\frac{1}{1\pm i}\exp\left[-\left(\frac{3|\omega|}{2}\right)^{\frac{1}{3}}(1\pm i)\,Y\right),$$

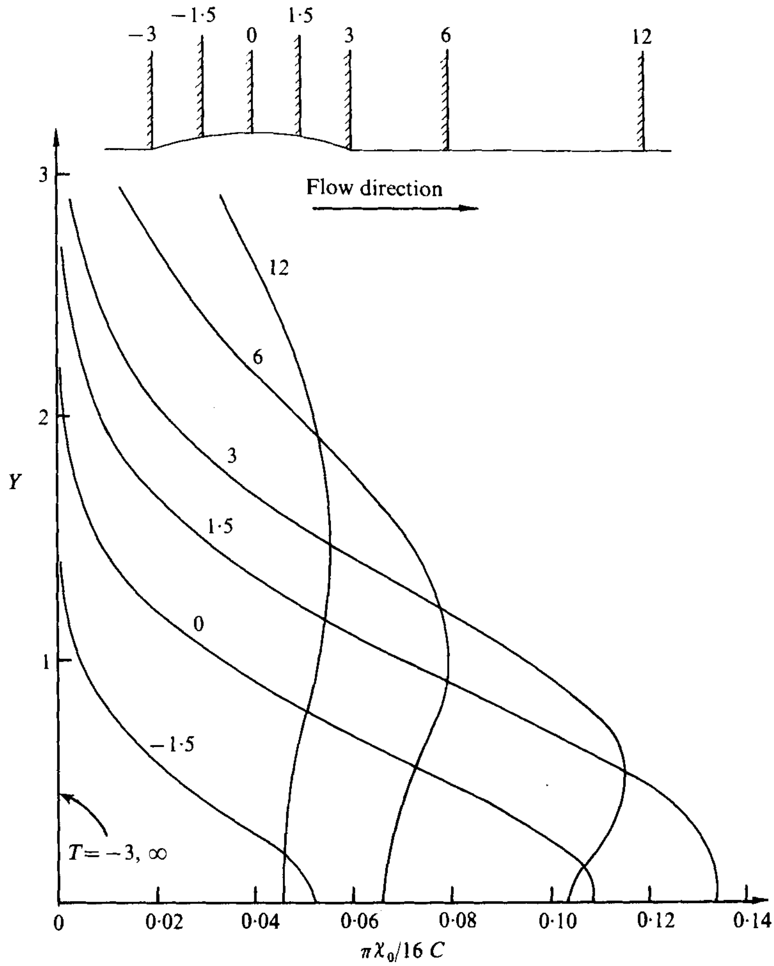

FIGURE 9. Deformations generated by an isolated bump on a side wall.

where the positive sign is appropriate when ω is positive and vice versa. Setting this equal to H_∞ at $S = 0$ yields

$$-\frac{1}{8}\frac{\partial^2\chi_0}{\partial Y^2}+\frac{3}{2}\frac{\partial\chi_0}{\partial T} = A\left(\frac{2}{3|\omega|}\right)^{\frac{1}{2}}\frac{1}{1\pm i}\exp\left[\frac{i\omega}{3}T-\left(\frac{3|\omega|}{2}\right)^{\frac{1}{2}}(1\pm i)\,Y\right], \tag{5.13}$$

and this can be used to describe the fluctuations in the position of the leading edge generated by an infinite wavy wall. Indeed, the time-periodic solution of (5.13) satisfying the condition

$$\partial\chi_0/\partial Y = 0 \quad \text{at} \quad Y = 0$$

is

$$\chi_0 = \frac{Ai(1\mp i)}{(2|\omega|)^{\frac{1}{2}}(\omega\mp\frac{3}{4}|\omega|)}\exp\left(-(2|\omega|)^{\frac{1}{2}}(1\pm i)\,Y+\frac{i\omega}{3}T\right)$$
$$-\frac{Ai(2/3|\omega|)^{\frac{1}{2}}(1\mp i)}{\omega\mp\frac{3}{4}|\omega|}\exp\left[-\left(\frac{3\omega}{2}\right)^{\frac{1}{2}}(1\pm i)\,Y+\frac{i\omega}{3}T\right].$$

Choosing $\omega = 1$ and taking the real part of the results gives

$$\chi_0 = \frac{4A}{2^{\frac{1}{2}}} \exp[-2^{\frac{1}{2}}Y][\cos(2^{\frac{1}{2}}Y - \tfrac{1}{3}T) + \sin(2^{\frac{1}{2}}Y - \tfrac{1}{3}T)]$$
$$-4A(\tfrac{2}{3})^{\frac{1}{2}} \exp[-(\tfrac{3}{2})^{\frac{1}{2}}Y]\{\cos[(\tfrac{3}{2})^{\frac{1}{2}}Y - \tfrac{1}{3}T] + \sin[(\tfrac{3}{2})^{\frac{1}{2}}Y - \tfrac{1}{3}T]\}, \quad (5.14)$$

where the wall displacement at the leading edge is

$$P_L = A\sin(\tfrac{1}{3}T). \quad (5.15)$$

Profiles for different values of T are sketched in figure 8.

Once the solution for a harmonic wall displacement has been obtained, it is a straightforward matter to construct the solution for an arbitrary wall displacement using Fourier synthesis. Figure 9 shows a sequence of profiles generated by flow past an isolated bump given by

$$P = \begin{cases} C(1-X^2), & |X| \leqslant 1, \\ 0, & |X| > 1. \end{cases}$$

Note that the disturbance dies out once the bump has been left behind by the leading edge.

The most striking feature of these results is that a wall displacement of slope $O(\epsilon)$ leads to leading-edge displacements with an $O(1)$ slope.

6. Concluding remarks

In this paper we have analysed four flows with the common feature that a geometrical length characteristic of the disturbance is much larger than a length characteristic of the depth divided by the bed slope. On a scale defined by the larger length the fluid depth is discontinuous at the leading edge; on the smaller scale the leading edge is quasi-one-dimensional. The solution of these problems has revealed extreme data sensitivity which can be summarized by the observation that, in practice, it will be very difficult to generate a flow with a straight leading edge. The analysis of § 2 shows that small inclinations of a straight channel to the fall line can create large deformations of the leading edge. In § 3 it is shown that small non-uniformities in the bed slope can generate leading-edge displacements that continue to grow long after the irregularities have been passed until eventually they become large. Similarly, § 5 reveals that slight unevenness in a side wall can generate large deformations. Thus although each of these flows is apparently stable in the sense that small initial disturbances eventually die out, their sensitivity can lead to effects similar to those associated with instability.

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Appendix. Uniqueness proof for the single-wall problem

It is sufficient to show that the solution to the following problem is identically zero:

$$\frac{\partial^2 H}{\partial Y^2} = 2\frac{\partial H}{\partial S} + 3\frac{\partial H}{\partial T};$$

$$\partial H/\partial Y = 0 \quad \text{at} \quad Y = 0; \quad H = 0 \quad \text{at} \quad T = 0;$$

$$H \to 0 \quad \text{as} \quad S \to -\infty; \quad H \to 0 \quad \text{as} \quad Y \to \infty.$$

Define the non-negative function $J(T)$ by

$$J = \int_0^\infty dY \int_{-\infty}^0 dS\, H^2.$$

Then

$$J'(T) = \frac{2}{3}\int_0^\infty dY \int_{-\infty}^0 dS\, H\left(\frac{\partial^2 H}{\partial Y^2} - 2\frac{\partial H}{\partial S}\right),$$

i.e.

$$J' = -\frac{2}{3}\int_0^\infty dY \int_{-\infty}^0 dS \left(\frac{\partial H}{\partial Y}\right)^2 - \frac{2}{3}\int_0^\infty dY\, H^2\bigg|_{S=0},$$

and so J is a non-increasing function. But J vanishes at $T = 0$ and so must be identically zero; and therefore H is identically zero.

REFERENCES

Dussan V., E. B. & Davis, S. H. 1974 *J. Fluid Mech.* **65**, 71–96.
Lin, S. P. 1974 *J. Fluid Mech.* **63**, 417–429.
Smith, P. 1969 *J. Engng Math.* **3**, 181–187.
Smith, S. H. 1969 *J. Engng Math.* **3**, 173–179.
Thompson, B. W. 1968 *J. Fluid Mech.* **31**, 379–395.